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# Effect of habitat complexity on rhinoceros and tiger population model with additional food and poaching in Kaziranga National Park, Assam

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## Abstract

This paper studies the effect of habitat complexity on Greater one-horned rhinoceros (Rhinoceros unicornis) and tiger (Panthera tigris) population model in Kaziranga National Park (KNP), Assam, India. Based on the analysis of the data collected from PCCF, Wildlife, Assam, three mathematical models are formulated and studied. In view of ecology, the main objective of the study is to increase the size of rhino population in aforesaid park. The mathematical stability and the complex dynamical behavior of systems are analyzed here. In KNP, the immature rhinos are killed by tiger, so the first system describes a stage structured prey–predator interaction , where the rhino is the prey and is divided into immature (horn not developed) and mature (horn developed) category and the tiger is the predator. The immature rhino is killed by the tiger following Holling type-II functional response, while the mature rhino falls prey to human induced poaching activities [Source: PCCF Wildlife, Assam, India]. Though Kaziranga is famous all over the world for the Greater one-horned rhinoceros, many other herbivores are also present in really good number and the tiger preys on them too. Thus, the previous system is modified by modifying the Holling type-II functional response incorporating the effect of additional food availability. To increase the number of rhino and to maintain the ecological balance of KNP, the second system is further extended by introducing habitat complexity in the Holling type-II functional response. In each system, equilibrium points have been obtained and their stability are discussed. Finally, numerical simulations are carried out to illustrate the analytical results. Based on the simulation results, it can be stated that the size of the rhino population increases in the presence of additional food and habitat complexity in spite of the poaching activities to a certain extent. The system shows complex dynamical behavior like Hopf bifurcation with respect to Poaching activity. Sensitivity analysis with respect to four important parameters viz., poaching effect, quality and quantity of the additional food and habitat complexity is also discussed.
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*Keywords:* Stage structure; Poaching; Additional food; Habitat complexity; Hopf-bifurcation

## 1. Introduction

Kaziranga National Park (KNP) is located between latitudes 26°34′N and 26°46′N and longitudes 93°80′E to 93°36′E within Nagaon, Golaghat and Sonitpur districts of Assam in the north-eastern part of India and including all the six new additions it covers a total area of about 858.98 sq. km. It is famous for the Greater one-horned rhinoceros

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(Rhinoceros unicornis) and is home to approximately 85% of the world population of this species [Source: Facts on Assam Forest (2018), Assam Forest Department] along with wide variety of other flora, fauna and avifauna including some of the endangered species. It has a very diverse and fragile ecosystem, where on one hand the annual flood water of the river Brahmaputra brings nourishment, leading to a very high productive biomass, and on the other hand, the phenomenon of erosion takes away lot of valuable and prime habitat. The park with its numerous habitat types ranging from the flood plains to the grasslands and evergreen forests provide several overlapping niches inside the park. The rhinos, the elephants, wild buffaloes, the swamp deer, the hog deer, tiger etc. are available here.

Some of the major threats to the wildlife of KNP are poaching, annual flood, erosion, siltation and weeds, livestock grazing and breach of embankment on eastern boundary of the park [Source: PCCF Wildlife, Assam, India]. Khushwaha et al. [11], Mathur et al. [20] and Medhi & Saha [21] discussed the land area change and rhino habitat suitability of the park and also noted the different challenges and problems of rhino habitat in KNP. Talukdar [41] had studied tiger predation of rhino calves at KNP. There are several studies on lake and other wetland ecosystems (Larsson [13], Scott [32], Ryding & Rast [27], Shukla & Dubey [36], Rai [26], Upadhyay et al. [43], Patra et al. [25]) where the effect of ecological changes on such ecosystems by various conditions are discussed and some solutions are recommended.

As we know the dynamics of predator–prey relationship is also recognized as an important factor in ecological system. Mathematical models relating to the dynamical behavior of the ecological systems, especially predator–prey systems have long been and continued to be widely applied in understanding the dynamical consequences of interacting populations, using theoretical and experimental approaches (Fay & Greeff [5], Lacitignola et al. [12], Sahoo & Poria [30], Samanta et al. [31], Dubey et al. [4]). In ecology, the dynamics of biological species depends on many factors, such as intra and inter-specific interference, death rate, stage structure, environmental conditions, habitat complexity, additional food, refuge region, time delay etc. It has been well established by Sih et al. [37] that the predator intensity is supposed to shape community structure and ecosystem properties. To understand the structure and function of a community, it is important to consider the impact of predation on prey population. Prey populations experience the effects of predation differently depending on how abundant the prey species is and, for actively foraging predators, how quickly the predator can find and consume prey (Hassell [7]). In nature, almost all animals have a stage structure in the form of immature and mature. The prey–predator system with stage structure for the prey, predator and for both considering various factors like harvesting, delay, refuge, diffusion has been widely studied in Liu & Zhang [17], Misra et al. [22], Shi et al. [35], Liu & Wang [16], Li & Xu [14], Kajanchi [8], Maiti et al. [19].

Again, the availability of additional food controls predator population and good quality of additional food favors rapid growth of predator population without affecting the prey population. Taylor [42] found that the risk of predation can be decreased by increasing the abundance of vulnerable prey therefore the additional food can protect the prey from the attack of predator. A predator–prey system when the predator is provided with some additional food was studied by Srinivasu et al. [38]. This study offers insight into the possible management strategies that involve manipulation of quality and supply level of additional food to predators, for the benefit of biological control. Sahoo & Poria [29] showed that increasing additional food for the top predator can control chaos in a three-species food chain model. Sahoo & Poria [28,30], Liu et al. [15], Sen et al. [33], Ghosh et al. [6], Srinivasu & Prasad [39], Sharma & Wightman [34], Kumar & Chakraborty [10] and other researches also considered additional food in their model system.

As predators subconsciously select habitats based on split-second decisions about distance-to-refuge, visual obstruction, and, down-weighted accordingly, forage availability, thus habitat complexity is one of the most intuitive concepts in ecology. According to Kovalenko et al. [9], the habitat complexity is the structural complexity of habitats. It can strongly mediate predator–prey interaction, affecting the total predation rates and its selectivity of different prey species or size classes. It also provides refuge to the prey from predation and thus high complexity areas are great natural protection. It decreases the interaction between prey and predator by reducing the searching efficiency of the predator. Many published works (Stoner [40], Bairagi & Jana [1,2], Pathak [24], Dubey et al. [4]) on predator–prey systems have incorporated habitat complexity with or without delay and studied effect of the degree of habitat complexity on the stability of a predator–prey model. Habitat complexity and gestation delay on the stability of a predator–prey model was formulated by Bairagi & Jana [1]. They found that the fluctuations in the population levels can be controlled completely by increasing the degree of habitat complexity. Again Bairagi & Jana [2] studied a predator–prey interaction in a homogeneously complex habitat with age structure on predator

population. Ma et al. [18] studied a generalized predator–prey system with habitat complexity in a different way and discussed destabilizing effect of habitat complexity. A two-dimensional prey–predator system with two delays, habitat complexity and prey refuge was studied by Dubey et al. [4]. The main objective of their work was to balance the prey–predator relationship in the presence of habitat complexity, prey refuge and delays.

The effects of supplying additional food to predator in a gestation delay induced predator–prey system with habitat complexity was analyzed by Sahoo & Poria [30]. They proved that the presence of additional food reduces the predator's attack rate on prey in the model and additional can control predator population also. Effects of prey refuge and additional food for predator in a diffusive predator–prey system was studied by Chakraborty et al. [3]. A predator–prey model with a prey refuge and additional food for predators was formulated by Samanta et al. [31]. They observed that enhancement of additional food for predators prevents the extinction of the predator and also replaces the stable limit cycle with a stable equilibrium.

In Kaziranga National Park, Assam, the immature rhino is killed by tiger population for food. The mature rhino falls prey to human induced poaching activity. The presence of additional food for the tiger in the form of other herbivores such as the Hog Deer (Axis porcinus), Sambar (Cervus unicolor), Barking Deer (Muntiacus muntjak), Swamp deer (Cervus duvaucelii), Wild Boar (Sun scrofa), Hog Badger (Arctonyx collaris), Wild Buffalo (Bubalus bubalis) in KNP increases the chances of survival of an immature rhino. Also, it is observed that the rhinos spend most of their time near the water bodies 'beels' surrounded by tall grasses making it difficult for the tiger to spot them. Again, the translocation of rhino (i.e. refuge factor) from KNP to the other forest has been carried out. Thus, from the definition of habitat complexity it is observed that these factors provide a refuge region for the immature rhino and decreases their interaction with the tiger. It is considered that the park may be defined to be in a state of ecological balance if it maintains higher densities of the rhino population compared to tiger population. But, if the densities of rhino or the tiger or both falls drastically it is ecologically imbalanced. The main aim of the present paper from ecological point of view is to increase the rhino population in the presence of poaching, additional food for the tiger, predation, infighting etc. so that the ecological balance of the park can be maintained. The complex dynamical behavior exhibited by the system is also examined.

To the best of author's knowledge, there have been no discussion yet on the age structured prey–predator system in presence of poaching effect, additional food and habitat complexity. Based on the analysis of real data, collected from PCCF Wildlife, Assam, a stage-structure rhinoceros–tiger population system on Kaziranga National Park is constructed. The interaction between immature Rhino and tiger population is assumed to be Holling type-II functional response. The poaching effect and intra-specific interference on mature Rhino is also included in the system. As the other animals like deer, buffalo etc. are also available in good number, thus the functional response is modified due to additional food in the second system. Again to increase the endangered animal rhino and to maintain the ecological balance of the park, the functional response is further modified by introducing habitat complexity in the third system. The organization of the paper is as follows. In Section 2, field data is analyzed to know the important factors. Section 3 is devoted to mathematical model formulation. The stability of systems I, II and III is performed in Sections 3, 4 and 5 respectively. Numerical simulation is carried out in Section 6. The discussion of the results is performed in Section 7.

## 2. Field data analysis

According to the census reports and data collected from PCCF Wildlife, Assam, KNP has seen tremendous growth in rhino population from meager figure of 12 nos. in 1908 to 2413 nos. in 2018 with a period of immense poaching during the 80's as shown in Fig. 2.1. Poaching of rhinos for its horn has always been a matter of concern for KNP. On an average 14 rhinos become victim of poaching activities every year. In spite of this, rhino population has improved to the extent that, translocation of rhinos from KNP to other forest has been carried out. Fig. 2.2 shows poaching and natural death of rhinos over the years. It is shown that the effect of poaching is large in (1981–1996, 2011–2017). Death of some mature rhinos has also mentioned in KNP due to infighting among themselves. The tigers in KNP kill on an average 12 immature rhinos every year even though they have ample food in the form of other animals. It is also observed from Figs. 2.1 and 2.2 that when the poaching effect was large during (1981–1996), there was fluctuation in the rhino population. Fig. 2.3 shows the tiger population over the years. The death of tiger in 2002–2009 is 42 (old age (12), infighting (10), knocked down by vehicles (4), poisoning by villagers (2), killed by buffalo (1), unknown (13)).
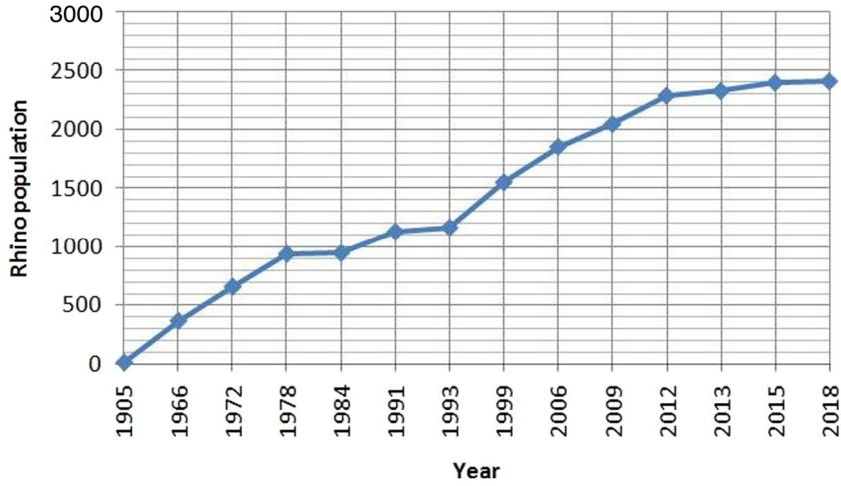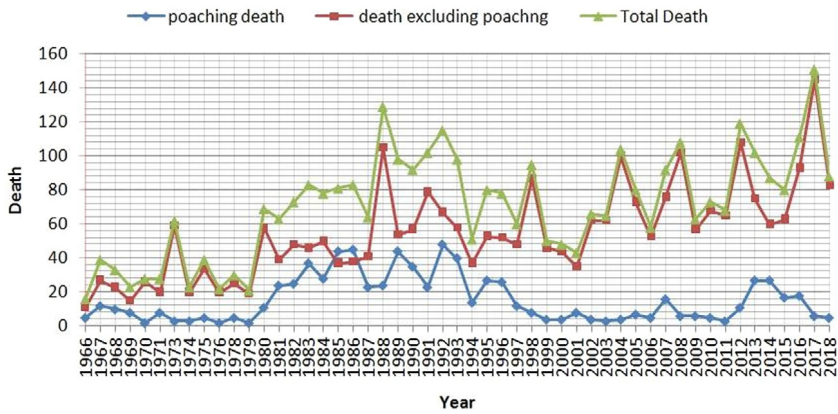
**Fig. 2.1.** Rhino population in KNP since 1905.



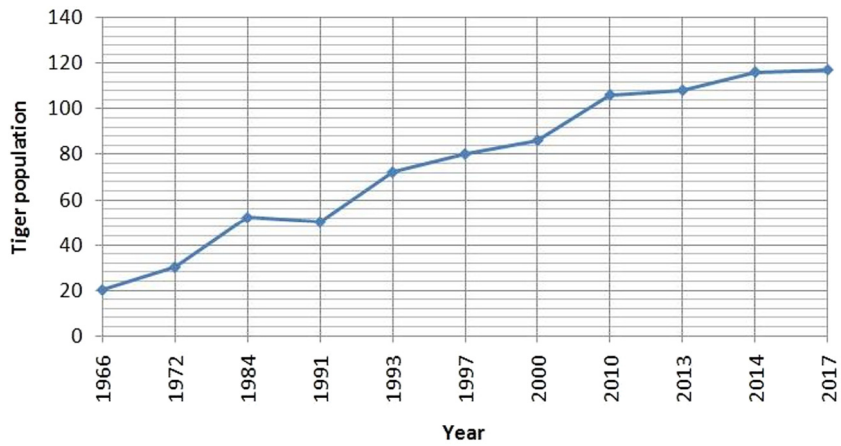**Fig. 2.2.** Total Rhino death in KNP due to poaching and natural causes.



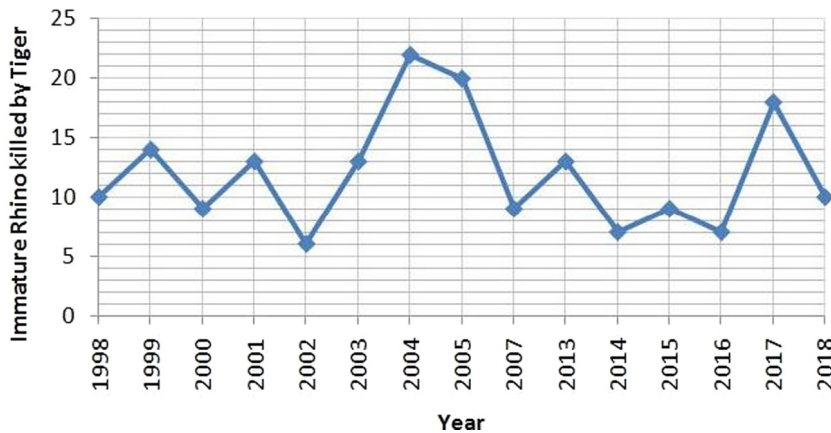**Fig. 2.3.** Tiger population in KNP.

**Fig. 2.4.** Immature Rhinos killed by Tiger in KNP.

Fig. 2.4 provides the information about the numbers of immature rhinos killed by tiger in KNP. It is large in 2002–2007 and 2016–2018. The reason might be the depletion of the additional food. Though, the census of other herbivores are not carried out as regularly by PCCF, Wildlife, Assam, but it is observed that during this period due to major flood and other factors less of these animals were available.

### 2.1. Necessary assumptions

Based on available data collected from the Department of Forest, Assam, regarding KNP, the following observations were made:

  i. The tigers kill immature rhino for food,
 ii. There is no interaction between mature rhino and tiger population,
iii. Poaching of mature rhino for its horn is a big issue,
 iv. Death of some mature rhino is also noticed due to infighting,
  v. Though KNP is famous for rhino, but other herbivores also exist in good number and the tiger preys on them too,
 vi. There is no inter-specific interference between rhino and other herbivores,
vii. The tall grasses around the water bodies "beels" provide a kind of refuge to the immature rhino,
viii. To increase the rhino population translocation of rhino from KNP to other forests has been carried out.

Considering the above facts this study proposes three mathematical models with the effects of including or excluding various phenomena present in population interactions in order to mimic the real situation in KNP as shown in Fig. 3.1.

In ecology, prey–predator interaction (i.e. functional response) is an important part for the model formulation. Basically, the predator functional response is defined as the amount of prey catch per predator per unit time. These functional responses are constructed in terms of behavior of an individual predator, which are routinely incorporated at the population level in systems that include reproduction and death. Some biological factors like handling time, encounter rate, searching efficiency, structure of prey population etc. affect the predator's functional response and accordingly the behavior of the dynamical system is diverted. Handling time may be defined as the amount of time it takes the predator to handle the food, beginning from the time the predator finds the prey item to the time the prey item is eaten. It decreases with the increase in prey density.

Holling type-I is a linear functional response. In this functional response the number of prey consumed by a predator per unit time followed by linear mass-action law and is given by

$$g(x) = ax \tag{2.1}$$

where $a(> 0)$ is capture rate. The feeding rate is proportional to the product of the densities for prey and predator populations. Mathematically, this becomes a straight line which passes through the origin. This is a traditional
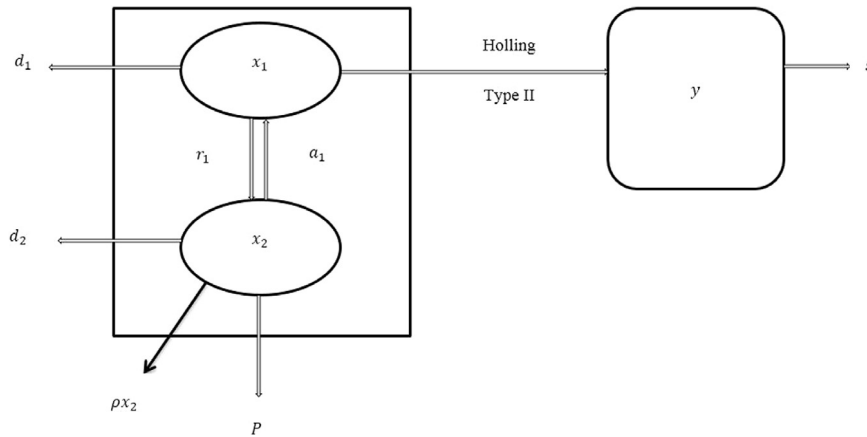
**Fig. 3.1.** Schematic diagram for the model system (3.2).

functional response with some drawbacks. If the availability of food will be high, predators will feed at maximum rate and hence further increase in food supply will not be able to increase the feeding rate further. It is also assumed that predator's handling time for prey is zero, it is observed that predators often become saturated in nature. To avoid the drawbacks, the most commonly used functional response Holling type-II is being considered here.

Holling type-II functional response describes the average feeding rate of predator when it spends some time for searching and some time for handling the prey. It is classified as prey dependent functional response. In this response, the number of prey consumed by a predator initially increases rapidly as the density of the prey increases, then it settles down at its steady state level. Thus the feeding rate of predator increases as prey density increases and then attains its maximum level.

This type of functional response is defined as:

$$g(x) = \frac{ax}{1 + ahx},$$
(2.2)

where $a$ is the attack rate and $h$ is the handling time.

Let the predator population be provided with additional food of constant biomass $M$, which is distributed uniformly in the habitat. It is assumed that the additional food is non-reproducing and is supplied at a constant rate. The number of additional food encounters per predator is proportional to the density of additional food. The proportionality constant characterizes the ability of the predator to identify the additional food (Srinivasu et al. [38]).

Based on the discussion of Sahoo & Poria [30] and Samanta et al. [31], let, $h_x$, $h_M$ denote tiger's handling time for rhino and additional food and $e_x$, $e_M$ represent tiger's searching efficiency for rhino and additional food respectively, then

$\mu = \frac{h_M}{h_x}$ indicates how long it takes the tiger to handle additional food relative to rhino population. This parameter characterizes quality of additional food. If $\mu > 1$ (i.e. $h_M > h_x$), then the tiger can easily capture immature rhino than additional food and it implies that the additional food available is of poor quality. And if, $\mu < 1$ then the additional food available is of rich quality.

$\eta = \frac{e_M}{e_x}$ indicates the ease with which tiger detects additional food relative to rhino population.

$A = \eta M$ represents the quantity of additional food for the tiger as it is directly proportional to the biomass of the additional food ($M$).

Thus the Holling type-II functional response with additional food $M$ is modified into the following form:

$$g(x) = \frac{ax}{1 + ahx + \mu A}.$$
(2.3)

Again, the existence of habitat complexity reduces the interaction rate between prey and the predator thereby reducing the probability of capturing prey by reducing the searching efficiency of the predator. Winfield [44] proved that the habitat complexity is more likely to affect the attack coefficient than the handling time. Thus the attack rate $a$ is replaced by $a(1 - c)$, where $c$ is a dimensional less quantity that measures the degree of habitat complexity

with $0 < c < 1$ and it reduces the predation rate. Again based on the discussion of Dubey et al. [4] and Sahoo & Poria [30], the Holling type-II functional response with habitat complexity is modified into the following form:

$$g(x) = \frac{a(1-c)x}{1 + a(1-c)hx + \mu A}. \tag{2.4}$$

It is to be noted, when $c = 0$ i.e. when there is no complexity and $\mu = 0$, we get back the Holling Type II response function.

The three systems are described below:

i. In the first system a stage structure for prey species (Rhinoceros unicornis) is considered. It is assumed that the tiger population feed on immature rhino, thus the interaction between tiger and immature rhino with Holling type-II functional response. The poaching effect and intra-specific interference on mature rhino are introduced in this system.

ii. To study the impact of predation on immature rhino in presence of additional food for predator, the Holling type-II functional response is modified in the second system. Similar to the first system there is no inter-specific interaction is noticed between rhino and other herbivores in this park.

iii. The second system is further extended to study the effect of habitat complexity on predation of immature rhino population.

## 3. The mathematical model formulation

Based on the discussion on field data presented in the previous section, some assumptions and other factors of importance for the age structured prey–predator systems are given below:

1. Let $x_1$, $x_2$, $y$ are the density of immature rhino, mature rhino and the tiger population at time $t$ respectively.
2. The birth rate of the immature rhino population is proportional to the existing mature rhino population with a proportionality constant $a_1$.
3. For the immature population, the death rate and transformation rate to mature are proportional to the existing immature population with proportionality constants $d_1$ and $r_1$ respectively.
4. The natural death rate of mature prey is assumed to be $d_2$.
5. The human poaching rate on mature prey is $P$.
6. Intra specific interference among mature rhinos is $\rho$.
7. According to the Forest Department of Assam there is no record of death due to in-fight between mature and immature rhino, so intra-specific competition between immature and mature rhino is not considered here.
8. Death rate of tiger population is $s$.
9. Coefficient $K$  ($0 < K < 1$) is the conversion efficiency, measuring the number of newly born predators for each captured prey.
10. Tiger as predator population do not kill mature rhino for food easily as for tiger other herbivores like swamp deer, buffalo are also available in good number. Only 4 such cases found during 2013–2018 [Source: PCCF Wildlife, Assam]; therefore in the model tiger and mature rhino interaction is not considered.
11. Let us assume that

$$a_1 r_1 > (d_1 + r_1)(d_2 + P), \tag{3.1}$$

i.e. the conversion rate of mature to immature rhino and birth rate of immature rhino is greater than a threshold value, which depends on death rate of mature and immature rhino and also on the rate of poaching. These assumption serve as foundation stone for the model formulation and is in conformity with the reality in the KNP.

### 3.1. Mathematical model-1

Here the rhino population is divided into immature and mature stage structure. The tiger feeds on immature rhino and it is followed by Holling Type -II functional response. The poaching effect and intra-specific interference on mature rhino are included in the system.

**Table 1**
Parameters and their unit.

| Parameter | Unit |
|---|---|
| $x_1$, $x_2$, $y$ | number area$^{-1}$ |
| $a_1$, $d_1$, $d_2$, $r_1$, $P$, $s$ | year$^{-1}$ |
| $a$, $\rho$ | area number$^{-1}$ year$^{-1}$ |
| $h$ | year |
| $K$ | *dimensionless constant* |

Thus, the system of equations is given below:

$$\frac{dx_1}{dt} = a_1 x_2 - d_1 x_1 - \frac{ax_1 y}{1 + ahx_1} - r_1 x_1,$$

$$\frac{dx_2}{dt} = r_1 x_1 - d_2 x_2 - P x_2 - \rho x_2^2, \tag{3.2}$$

$$\frac{dy}{dt} = -sy + K \frac{ax_1 y}{1 + ahx_1},$$

with initial conditions $x_1(0) > 0$, $x_2(0) > 0$, $y(0) > 0$ and parameter units as shown in Table 1.

### 3.2. Positivity and boundedness of solutions

The positivity and boundedness of a system are very important. Positivity guarantees that the system is well posed and densities cannot become negative and the boundedness represents growth restriction as a consequence of limited resources.

**Positive Invariance:**

The system (3.2) can be put in the matrix form as

$$\dot{\bar{X}} = F(\bar{X}) \text{ with } \bar{X}(0) = \bar{X}_0 \in R_+^3, \text{ where } \bar{X} = (x_1, x_2, y)^T \in R_+^3,$$

$$F(\bar{X}) = \begin{bmatrix} a_1 x_2 - d_1 x_1 - \frac{ax_1 y}{1+ahx_1} - r_1 x_1 \\ r_1 x_1 - d_2 x_2 - P x_2 - \rho x_2^2 \\ \frac{Kax_1 y}{1+ahx_1} - sy \end{bmatrix},$$

$$F: R^3 \to R_+^3 \text{ and } F \in C^\infty(R^3).$$

It can be seen that whenever $\bar{X}(0) \in R_+^3$ such that $X_i = 0$ then $F_i(\bar{X})\big|_{x_i=0} \geq 0$ for (i = 1,2,3). Now any solution of $\dot{\bar{X}} = F(\bar{X})$ with $\bar{X}_0 \in R_+^3$, say $\bar{X}(t) = \bar{X}(t, \bar{X}_0)$ is such that $\bar{X}(t) \in R_+^3$ for all $t > 0$ (Nagumo [23]).

**Boundedness:**

**Lemma 3.1.** *The set* $\Omega_1 = \{(x_1, x_2, y): 0 < x_1 \leq \frac{a_1^2}{4\rho v}, \ 0 < x_2 \leq \frac{a_1^2}{4\rho v}, \ 0 < y \leq \frac{1}{K}\frac{a_1^2}{4\rho v}, \ 0 < x_1(t)+x_2(t)+\frac{y(t)}{K} \leq \frac{a_1^2}{4\rho v}\}$ *attracts all solutions in the interior of the positive orthant, where,* $v = min\{d_1, d_2 + P, \frac{s}{K}\}.$

**Proof.** Let us consider a time dependent function

$$W(t) = x_1(t) + x_2(t) + \frac{1}{K} y(t).$$

Clearly

$$\frac{dW}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} + \frac{1}{K}\frac{dy}{dt}$$

$$= -\{d_1 x_1 + (d_2 + P)x_2 + \frac{s}{K}y\} + (a_1 x_2 - \rho x_2^2).$$

Taking $v = min\{d_1, d_2 + P, \frac{s}{K}\}$ we get

$$\frac{dW}{dt} + vW \leq \frac{a_1^2}{4\rho} \quad \text{so} \quad W \leq \frac{a_1^2}{4\rho v} + \left(W(0) - \frac{a_1^2}{4\rho v}\right)e^{-vt}.$$

As $t \to \infty$, $\quad W \leq \frac{a_1^2}{4\rho v}$.

Therefore all solutions of system (3.2) initiating from any point in $R_+^3$ tends to (as $t \to \infty$) a fixed bounded region $\Omega_1$. Thus the system is ecologically well behaved. □

To see the behavior of the system, the equilibrium points will be determined first, then the variational matrix will be found at each equilibrium points.

### 3.3. Existence of equilibrium points

System (3.2) possesses the following equilibrium points:

i. $E_0(0, 0, 0)$ always exists.
ii. $E_1(\bar{x}_1, \bar{x}_2, 0)$ exists under the condition (3.1), where,

$$\bar{x}_1 = \frac{a_1 \bar{x}_2}{d_1 + r_1},$$
$$\bar{x}_2 = \frac{a_1 r_1 - (d_2 + P)(d_1 + r_1)}{(d_1 + r_1)\rho}.$$

iii. From ecological and biological point of view, it is important to determine the uniqueness of the positive equilibrium point $E_2(x_1^*, x_2^*, y^*)$, which exists, provided the following conditions are satisfied

$$(i) \quad a_1 > 2\rho\left[\frac{(d_1 + r_1)\frac{s}{a(K-sh)}}{\left\{-(d_2 + P) + \sqrt{(d_2 + P)^2 + \frac{4\rho r_1 s}{a(K-sh)}}\right\}}\right], \tag{3.3}$$

i.e. birth rate of immature rhino is greater than a threshold value.

$$(ii) \quad K > sh,$$

where,

$$x_1^* = \frac{s}{a(K - sh)},$$
$$x_2^* = \frac{-(d_2 + P) + \sqrt{(d_2 + P)^2 + \frac{4r_1 s\rho}{a(K-sh)}}}{2\rho},$$
$$y^* = \frac{K}{s}\left\{a_1 x_2^* - \frac{s(d_1 + r_1)}{a(K - sh)}\right\}.$$

In the following section the stability of the equilibrium points are analyzed.

### 3.4. Local stability analysis

For the system (3.2), the stability of the equilibrium points can be determined by the nature of the eigen values of the Jacobian matrix evaluated at the corresponding equilibrium points using Routh–Hurwitz criterion. The results are given below:

i. $E_0(0, 0, 0)$ is always a saddle point.
ii. (a) $E_1(\bar{x}_1, \bar{x}_2, 0)$ is locally asymptotically stable , if

$$\bar{x}_1 < \frac{s}{a(K - sh)}.$$

(b) Otherwise $E_1(\bar{x}_1, \bar{x}_2, 0)$ unstable.

iii. The interior equilibrium point $E_2(x_1^*, x_2^*, y^*)$ is locally asymptotically stable iff

$$\left\{ d_1 + r_1 + \frac{ay^*}{(1 + ahx_1^*)^2} \right\}(d_2 + P + 2\rho x_2^*) > a_1 r_1, \tag{3.4}$$

i.e. the birth rate of the immature rhino and transformation rate to mature rhino is less than a threshold value.

### 3.5. Hopf-bifurcation

Considering poaching effect $P$ as a bifurcation parameter, the necessary and sufficient conditions for Hopf-bifurcation to occur are

$$
\begin{array}{ll}
(i) & A_1(P^*) > 0, \ A_3(P^*) > 0, \\
(ii) & f(P^*) = A_1(P^*)A_2(P^*) - A_3(P^*) = 0 \ \ \text{and} \\
(iii) & Re\left[\dfrac{d\lambda_j}{dP}\right]_{P=P^*} \neq 0, \ j = 1, 2, 3.
\end{array}
\tag{3.5}
$$

where $P = P^*$ is the critical value of $P$. $A_1$, $A_2$ and $A_3$ are the coefficients in the characteristic equation of the variational matrix evaluated at $E_2(x_1^*, x_2^*, y^*)$ given by

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{3.6}$$

with

$$A_1 = d_1 + r_1 + \frac{ay^*}{(1 + ahx_1^*)^2} + d_2 + P + 2\rho x_2^* > 0,$$

$$A_2 = (d_1 + r_1 + \frac{ay^*}{(1 + ahx_1^*)^2})(d_2 + P + 2\rho x_2^*) + \frac{Ka^2 x_1^* y^*}{(1 + ahx_1^*)^3} - a_1 r_1,$$

$$A_3 = \frac{Ka^2 x_1^* y^*}{(1 + ahx_1^*)^3}(d_2 + P + 2\rho x_2^*) > 0,$$

and $\lambda_j$ is the eigen value of the variational matrix associated with $E_2$.

From the second condition $f \equiv A_1 A_2 - A_3 = 0$, we get an equation $P$, which has at least one root $P^*$ (say). For some $\epsilon > 0$, there exist an interval $(P^* - \epsilon, P^* + \epsilon)$ for which $P^* - \epsilon > 0$, so that $A_2 > 0$ for $P \in (P^* - \epsilon, P^* + \epsilon)$. Thus the characteristic equation of $E_2$ cannot have any real positive root for $P \in (P^* - \epsilon, P^* + \epsilon)$.

So , for $P = P^*$ Eq. (3.6) can be written as

$$(\lambda^2 + A_2)(\lambda^2 + A_1) = 0 \tag{3.7}$$

This has three roots i.e. $\lambda_{1,2} = \pm i\omega, \ \ \lambda_3 = \delta$ where $\omega = \sqrt{A_2}$ and $\delta = -A_1$. For $P \in (P^* - \epsilon, P^* + \epsilon)$, these roots can be taken in the following form

$$
\begin{aligned}
\lambda_1 &= \mu_1(P) + i\mu_2(P) \\
\lambda_2 &= \mu_1(P) - i\mu_2(P) \\
\lambda_3 &= -A_1(P)
\end{aligned}
$$

Now, to verify the third condition of Eq. (3.5), differentiating Eq. (3.6) w.r.t $P$, we have

$$3\lambda^2 \frac{d\lambda}{dP} + \dot{A}_1 \lambda^2 + 2A_1 \lambda \frac{d\lambda}{dP} + \dot{A}_2 \lambda + A_2 \frac{d\lambda}{dP} + \dot{A}_3 = 0$$

$$\Rightarrow \frac{d\lambda}{dP} = -\frac{\dot{A}_1 \lambda^2 + \dot{A}_2 \lambda + \dot{A}_3}{(3\lambda^2 + 2A_1 \lambda + A_2)}\Bigg|_{\lambda = i\sqrt{A_2}}$$

$$= \frac{\dot{A}_3 - (\dot{A}_1 A_2 + \dot{A}_2 A_1)}{2(A_1^2 + A_2)} + i\frac{\sqrt{A_2}(\dot{A}_2 A_2 + A_1 \dot{A}_3 - \dot{A}_1 A_2 A_1)}{2A_2(A_1^2 + A_2)}$$

Thus $\frac{d}{dP}[Re(\lambda)]_{P=P^*} = -\frac{\frac{dR}{d\lambda}}{2(A_1^2 + A_2)} \neq 0$, $R = A_1 A_2 - A_3$.

Thus, the following theorem can be stated as

**Theorem 3.1.** *There is a Hopf-bifurcation at equilibrium point $E_2$ under the condition* (3.5) *for the critical value* $P = P^*$.

*3.6. Global stability*

In this section the global stability of the system of Eq. (3.2) is discussed.

**Theorem 3.2.** *If the following inequalities hold in the region $\Omega_1$ :*

$$(a) \quad (d_1 + r_1)(1 + ahx_1^*) > a^2 hx_1^* y^*,$$

$$(b) \quad 2\left(d_1 + r_1 - \frac{a^2 hx_1^* y^*}{1 + ahx_1^*}\right)\left(\frac{4r_1 x_1^* v}{x_2^* a_1^2} + 1\right) > \frac{a_1^2}{\rho}, \tag{3.8}$$

*then the positive equilibrium point $E_2$ is globally asymptotically stable.*

**Proof.** Define a positive definite function $V_1$ about $E_2$, such that

$$V_1(x_1, x_2, y) = (x_1 - x_1^*)^2 + (x_2 - x_2^* - \ln \frac{x_2}{x_2^*}) + l(y - y^* - \ln \frac{y}{y^*}),$$

where, $l$ is a suitable constant to be determined. Differentiating $V_1$ with respect to $t$ along the solution of the system (3.2) , we get,

$$\frac{dV_1}{dt} = -a_{11}(x_1 - x_1^*)^2 + a_{12}(x_1 - x_1^*)(x_2 - x_2^*) - a_{22}(x_2 - x_2^*)^2 + a_{13}(x_1 - x_1^*)(y - y^*),$$

where,

$$a_{11} = 2\left\{d_1 + r_1 + \frac{ay}{1 + ahx_1} - \frac{a^2 hx_1^* y^*}{(1 + ahx_1)(1 + ahx_1^*)}\right\},$$

$$a_{12} = 2a_1 + \frac{r_1}{x_2},$$

$$a_{22} = \rho + \frac{r_1 x_1^*}{x_2 x_2^*}, \tag{3.9}$$

$$a_{13} = \left\{\frac{lKa}{1 + ahx_1^*} - 2ax_1^*\right\}\frac{1}{1 + ahx_1}.$$

Choose $l = (1 + ahx_1^*)\frac{2x_1^*}{K}$, then $a_{13} = 0$.

Thus the sufficient condition for $\frac{dV_1}{dt}$ to be negative definite is that the following conditions hold:

$$a_{11} > 0 \text{ and } a_{12}^2 < 4a_{11}a_{22} \tag{3.10}$$

and (3.10) is satisfied under the condition of (3.8). Thus '$V_1$' is a Lyapunov function with respect to all solutions initiating in the interior of the positive orthant.  □

**Note:** This is to be noted that the conditions of local and global stability are different. So, if $E_2$ is locally asymptotically stable, it is not necessary that it will also be globally asymptotically stable. But if $E_2$ is globally asymptotically stable in a region $\Omega_1$ containing the point $E_2$, then it is stable locally also.

## 4. Mathematical model-II

From the data collected from PCCF Wildlife, Assam, it was observed that the tiger population preys not only on the immature rhino but also on other herbivores of KNP. The previous system (Mathematical model-I) was formulated by considering the rhino as the main food for tiger. In this section, the other herbivores are considered as "additional food" for tiger population and accordingly, the system (3.2) is modified by modifying the functional

response to make way for availability of additional food for the predator. Thus, considering Sahoo & Poria [30], Samanta et al. [31] and the functional response (2.2) the system of equations become

$$\frac{dx_1}{dt} = a_1 x_2 - d_1 x_1 - \frac{ax_1 y}{1 + \mu A + ahx_1} - r_1 x_1,$$

$$\frac{dx_2}{dt} = r_1 x_1 - d_2 x_2 - Px_2 - \rho x_2^2, \tag{4.1}$$

$$\frac{dy}{dt} = -sy + \frac{Kax_1 y}{1 + \mu A + ahx_1} + \frac{K\mu y}{1 + \mu A + ahx_1},$$

with initial conditions $x_1(0) > 0$, $x_2(0) > 0$ and $y(0) > 0$. Here, the unit of the product of parameters relating to the quality $\mu$ and quantity $A$ of the additional food is dimensionless.

In the following lemma, we show that all solutions of system (4.1) are bounded which implies that the system is ecologically well behaved.

**Lemma 4.1.** *The set* $\Omega_2 = \left\{ (x_1, x_2, y) \colon 0 < x_1 \leq \frac{a_1^2}{4\rho d_0},\ 0 < x_2 \leq \frac{a_1^2}{4\rho d_0},\ 0 < y \leq \frac{1}{K} \frac{a_1^2}{4\rho d_0},\ 0 < x_1(t) + x_2(t) + \frac{y(t)}{K} \leq \right.$ $\left. \frac{a_1^2}{4\rho d_0} \right\}$ *attracts all solutions in the interior of the positive orthant, where* $d_0 = min\{d_1,\ d_2 + P,\ s - \frac{K\mu}{1 + \mu A}\}$ *and* $s > \frac{K\mu}{(1 + \mu A)}$.

The proof is similar to Lemma 3.1, thus omitted.
In the following section we analyze the system (4.1).

### 4.1. Existence of equilibrium points

The system (4.1) has three non-negative equilibria as shown below.

  i. $O_0(0, 0, 0)$ always exists.
 ii. $O_1(\tilde{x}_1, \tilde{x}_2, 0)$ exists under the condition (3.1), where $\tilde{x}_1$, $\tilde{x}_2$ are same as $\bar{x}_1$, $\bar{x}_2$ of the system (3.2).
iii. $O_2(\ddot{x}_1, \ddot{x}_2, \ddot{y})$ exists if

$$s(1 + \mu A) > K\mu \text{ and } K > sh, \tag{4.2}$$

where,

$$\ddot{x}_1 = \frac{s(1 + \mu A) - K\mu}{a(K - sh)},$$

$$\ddot{x}_2 = \frac{-(d_2 + P) + \sqrt{(d_2 + P)^2 + 4r_1 \rho \left\{ \frac{s(1 + \mu A) - K\mu}{a(K - sh)} \right\}}}{2\rho},$$

$$\ddot{y} = \frac{K\{1 + \mu(A - h)\}}{\{s(1 + \mu A) - K\mu\}} [a_1 \ddot{x}_2 - (d_1 + r_1)\ddot{x}_1].$$

### 4.2. Stability analysis

Here the local as well as global stability of the system (4.1) around the equilibrium points are discussed. The results are given below:

  i. The equilibrium point $O_0(0, 0, 0)$ is always a saddle point.
 ii. The equilibrium point $O_1(\tilde{x}_1, \tilde{x}_2, 0)$ is

    (a) locally asymptotically stable if $s > \frac{K(a\tilde{x}_1 + \mu)}{1 + ah\tilde{x}_1 + \mu A}$,

    (b) otherwise it is a saddle point.

iii. The interior equilibrium point $O_2(\ddot{x}_1, \ddot{x}_2, \ddot{y})$ is locally asymptotically stable iff

$$\left\{ d_1 + r_1 + \frac{a\ddot{y}(1 + \mu A)}{(1 + \mu A + ah\ddot{x}_1)^2} \right\} (d_2 + P + 2\rho\ddot{x}_2) > a_1 r_1. \tag{4.3}$$

iv. The positive equilibrium point $O_2$ is globally asymptotically stable in the region $\Omega_2$ under the following conditions:

$$
\begin{align}
(a) \quad & (d_1 + r_1)(1 + \mu A)(1 + ah\ddot{x}_1 + \mu A) > a^2 h\ddot{x}_1\ddot{y}, \\
(b) \quad & 2\left\{d_1 + r_1 - \frac{a^2 h\ddot{x}_1\ddot{y}}{(1 + \mu A)(1 + ah\ddot{x}_1 + \mu A)}\right\}\left\{1 + \frac{4d_0\ddot{x}_1 r_1}{\ddot{x}_2 a_1^2}\right\} > \frac{a_1^2}{\rho}.
\end{align}
\tag{4.4}
$$

**Proof.** The positive definite function about $O_2$ is

$$
V_2(x_1, x_2, y) = (x_1 - \ddot{x}_1)^2 + (x_2 - \ddot{x}_2 - \ln\frac{x_2}{\ddot{x}_2}) + l_1(y - \ddot{y} - \ln\frac{y}{\ddot{y}}),
$$

where, $l_1$ is a suitable constant to be determined. The proof is similar to Theorem (3.2), thus omitted. $\quad\square$

**Note:** The conditions (4.4) are modified version of conditions (3.8) with the modified term quality and quantity of the additional food.

## 5. Mathematical model-III

In the previous section, presence of additional food was considered to save the immature rhino from tiger population. At the same time, to increase the immature rhino population specific effort is needed. If, the rhino is kept in a reserved region where the rhino tiger interaction is less and poaching is greatly reduced then, the rhino population may eventually increase. Keeping all these in mind, the habitat complexity is introduced in the system and accordingly, the Holling type-II functional response followed by Dubey et al. [4], Sahoo& Poria [30] is modified. Thus considering the modified Holling type-II functional response (i.e. Eq. (2.3)), the system of equations become:

$$
\begin{align}
\frac{dx_1}{dt} &= a_1 x_2 - d_1 x_1 - \frac{a(1-c)x_1 y}{1 + \mu A + a(1-c)hx_1} - r_1 x_1, \\
\frac{dx_2}{dt} &= r_1 x_1 - d_2 x_2 - P x_2 - \rho x_2^2, \\
\frac{dy}{dt} &= -sy + \frac{Ka(1-c)x_1 y}{1 + \mu A + a(1-c)hx_1} + \frac{K\mu y}{1 + \mu A + a(1-c)hx_1},
\end{align}
\tag{5.1}
$$

with initial conditions $x_1(0) > 0$, $x_2(0) > 0$ and $y(0) > 0$, where $c$ $(0 < c < 1)$ is a dimensionless constant representing the degree of habitat complexity.

In the following Theorem, we show that all solutions of system (5.1) are bounded.

**Lemma 5.1.** *The set* $\Omega_3 = \Big\{ (x_1, x_2, y): 0 < x_1 \le \frac{a_1^2}{4\rho\gamma},\ 0 < x_2 \le \frac{a_1^2}{4\rho\gamma},\ 0 < y \le \frac{1}{K}\frac{a_1^2}{4\rho\gamma}$ *and* $0 < x_1(t) + x_2(t) + \frac{y(t)}{K} \le \frac{a_1^2}{4\rho\gamma} \Big\}$ *attracts all solutions in the interior of the positive orthant, where* $\gamma = min\{d_1,\ d_2 + P,\ s - \frac{K\mu}{1+\mu A}\}$ *and* $s > \frac{K\mu}{1+\mu A}$.

The proof of Lemma 5.1 can be worked out similar to Lemma 3.1.
In the following section we analyze the system (5.1).

### 5.1. Existence of equilibrium points

The system (5.1) has three non-negative equilibria as shown below.

　i. $N_0(0, 0, 0)$ always exists.
　ii. $N_1(\check{x}_1, \check{x}_2, 0)$, exists under the condition of (3.1).

iii. $N_2(\widehat{x}_1, \widehat{x}_2, \widehat{y})$ exists under the condition same as (4.2) where,

$$
\begin{aligned}
\widehat{x}_1 &= \frac{s(1 + \mu A) - K\mu}{a(K - sh)(1 - c)}, \\
\widehat{x}_2 &= \frac{-(d_2 + P) + \sqrt{(d_2 + P)^2 + 4\widehat{x}_1 \rho r_1}}{2\rho}, \\
\widehat{y} &= \frac{K\{1 + \mu(A - h)\}}{\{s(1 + \mu A) - K\mu\}}[a_1 \widehat{x}_2 - (d_1 + r_1)\widehat{x}_1].
\end{aligned}
\tag{5.2}
$$

### 5.2. Stability analysis

Here the local as well as global stability of the system (5.1) around the equilibrium points are discussed. The results are given below:

   i. The equilibrium point $N_0(0, 0, 0)$ is always saddle.

  ii. The equilibrium point $N_1(\check{x}_1, \check{x}_2, 0)$ is

      (a) stable if $s > \frac{K[a(1-c)\check{x}_1 + \mu]}{1 + ah(1-c)\check{x}_1 + \mu A}$,

      (b) otherwise it is unstable.

  iii. The interior equilibrium point $N_2(\widehat{x}_1, \widehat{x}_2, \widehat{y})$ is locally asymptotically stable iff

$$
a_1 r_1 < \left\{ d_1 + r_1 + \frac{a(1 - c)(1 + \mu A)\widehat{y}}{(1 + \mu A + a(1 - c)h\widehat{x}_1)^2} \right\}(d_2 + P + 2\rho\widehat{x}_2).
\tag{5.3}
$$

  iv. If the following conditions hold true

      $(a)$   $(d_1 + r_1)(1 + \mu A)\{1 + ah(1 - c)\widehat{x}_1 + \mu A\} > a^2(1 - c)^2 h\widehat{x}_1\widehat{y},$

$$
(b)\quad 2\left\{ d_1 + r_1 - \frac{a^2(1 - c)^2 h\widehat{x}_1\widehat{y}}{(1 + \mu A)(1 + ah(1 - c)\widehat{x}_1 + \mu A)} \right\}(\frac{4r_1\widehat{x}_1\gamma}{\widehat{x}_2 a_1^2} + 1) > \frac{a_1^2}{\rho}
\tag{5.4}
$$

    then $N_2$ is globally asymptotically stable in the region $\Omega_3$.

**Proof.** The positive definite function about $N_2$ is

$$
V_3(x_1, x_2, y) = (x_1 - \widehat{x}_1)^2 + (x_2 - \widehat{x}_2 - \ln \frac{x_2}{\widehat{x}_2}) + l_2(y - \widehat{y} - \ln \frac{y}{\widehat{y}}),
$$

where, $l_2$ is a suitable constant to be determined. The proof is similar to Theorem (3.2), thus omitted.   □

**Note:** The conditions of global stability (5.4) are same as conditions (4.4) with extra term habitat complexity.

## 6. Numerical simulation

Numerical simulations are carried out in this section to validate the analytical results that were obtained in the previous sections. The following parameters namely poaching rate '$P$', the quality '$\mu$', quantity '$A$' of the additional food and habitat complexity of the immature rhino '$c$' are the four key parameters that directly influence the dynamics of the system and ecologically balanced behavior of the park. This section is divided into three parts based on:

   i. The ecological behavior of the park ,

  ii. The complex dynamical behavior of the system, and

  iii. Sensitivity analysis.

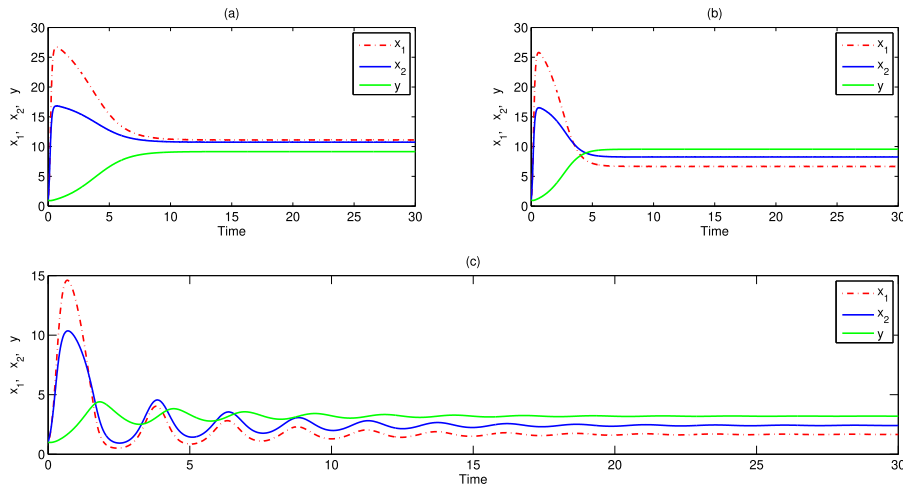For this some biologically feasible parameters are chosen that satisfies various analytical conditions.

**Fig. 6.1.** Ecologically balanced and imbalanced behavior of the park w.r.t $P$.

### 6.1. Ecological behavior of KNP

To see the ecological behavior of the park and to observe the effect of key parameters ( poaching rate '$P$', the quality '$\mu$', quantity '$A$' of the additional food and habitat complexity of the immature rhino '$c$' ) on ecological balance of KNP, the following set of values of parameters are chosen:

$$a_1 = 45, \quad d_1 = 5, \quad a = 3, \quad h = 0.02, \quad r_1 = 22, \quad \rho = 2,$$
$$d_2 = 0.3, \quad P = 1, \quad s = 1, \quad K = 0.05, \quad \mu = 0, \quad A = 0 \qquad (6.1)$$

with the initial conditions $x_1(0) = 1$, $x_2(0) = 1$ and $y(0) = 1$. These values of parameters satisfy the local stability condition (Eq. (3.4)) of the system (3.1). Thus the positive equilibrium point is locally asymptotically stable. Now, we will concentrate our attention to maintain the ecological balanced behavior of the park.

#### 6.1.1. Behavior of the park for the system (3.1)

For the set of values of parameters given in Eq. (6.1), it is observed that when attacking rate by the tiger population ($a = 3$) and poaching effect ($P = 1$) is less, then the density of rhino population is greater than tiger population (Fig. 6.1(a)). But if the attacking rate ($a = 5$) increases, then the density of tiger population also increases compared to rhino population (Fig. 6.1(b)). If the attacking rate as well as poaching effect increase ($a = 20$, $P = 10$), then the trajectories first oscillate and then go to the respective equilibrium level. Here the density of tiger population is larger than rhino population (Fig. 6.1(c)). Thus the park is in an ecologically imbalanced situation. To recover from this situation, the additional food is introduced in the second system so that attacking rate can be minimized. Now, the role of quality and quantity of the additional food is observed to maintain the ecological balance of the park.

#### 6.1.2. Behavior of the park for the system (4.1)

As mentioned earlier, $\mu$ indicates the quality of the additional food. Thus $\mu$ is divided into two parts : $\mu > 1$ i.e. additional food is of poor quality and $\mu < 1$, i.e. the additional food is of high quality. Then for different values of $A$, the behavior of immature rhino, mature rhino and tiger population with respect to time $t$ are given in Figs. 6.2 and 6.3. Thus this section is divided into two subsections:

i. $\mu > 1$ i.e. when the additional food is of poor quality.
   For the values of parameters (6.1) with $a = 20$, $P = 10$, $\mu = 10$, $A = 0.2$, it is observed from (Fig. 6.2(a)) that the tiger population settles in the higher equilibrium level whereas the rhino go to the lower equilibrium level. Thus the park is not showing ecologically balanced behavior. But if the value of $A$ increases i.e. $A = 0.5$ (Fig. 6.2(b)), the density of rhino population increases and goes to a higher equilibrium level compared to the density of tiger population. So, the park is showing ecologically balanced behavior.
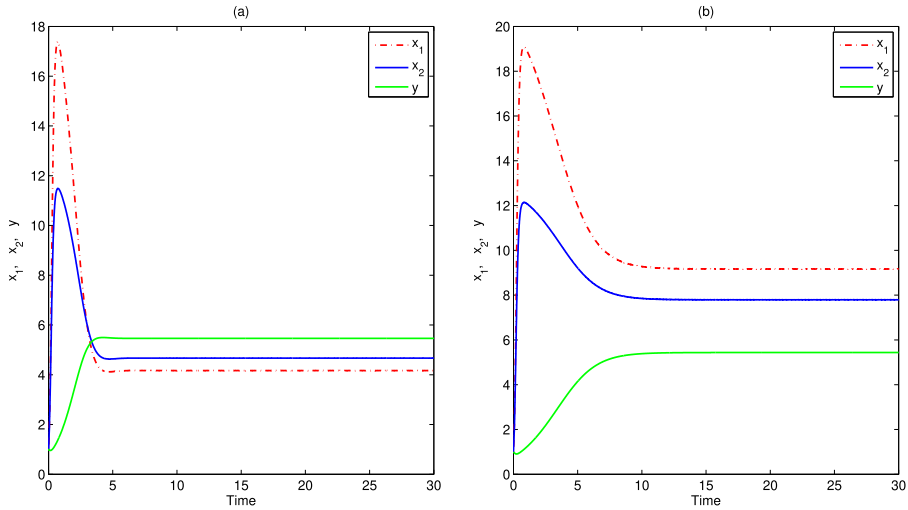
**Fig. 6.2.** Ecologically imbalanced and balanced behavior of the system (4.1) w.r.t $A$ for $\mu > 1$.
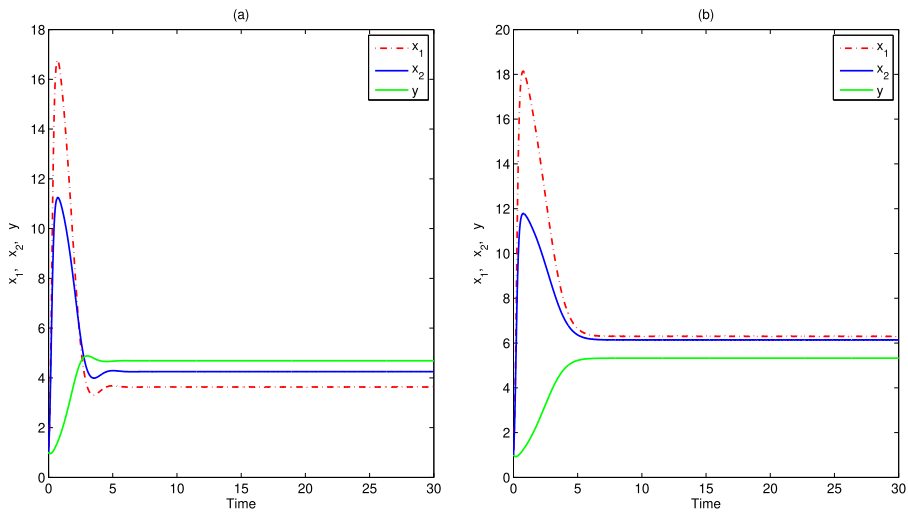


**Fig. 6.3.** Ecologically imbalanced and balanced behavior of the system (4.1) w.r.t $A$ for $\mu < 1$.

ii. $\mu < 1$ i.e. when the additional food is of high quality.

Choosing the same set of values of parameters (6.1) with $a = 20$, $P = 10$, $\mu = .4$, $A = 3$, it is observed from (Fig. 6.3(a)) that the density of the tiger population is higher than densities of rhino population. Thus the park is in an ecologically imbalanced situation. But the behavior changes if the value of $A$ increases (i.e. for $A = 5$, (Fig. 6.3(b)).

Thus the balanced behavior of the park can be maintained by increasing the quantity of the additional food.

But to maintain the balanced behavior of the park in the presence of poor or high quality with poor quantity of the additional food, high attacking rate, poaching effect, some effort is needed to recover this situation. Thus the habitat complexity is added to the system (4.1).
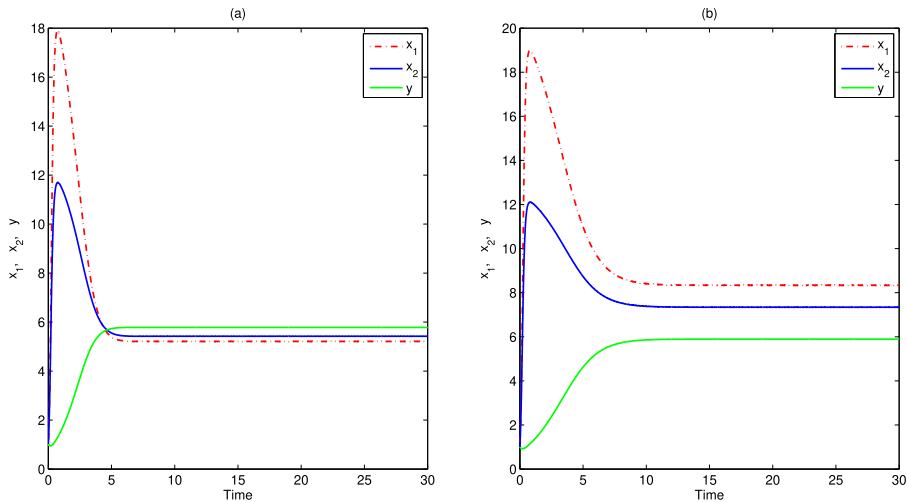
**Fig. 6.4.** Ecologically imbalanced and balanced behavior of the system (5.1) w.r.t $c$ for $\mu > 1$.

### 6.1.3. Behavior of the park for the system (5.1)

This section is also divided into two subsections.

   i. When $\mu > 1$

For set of values of parameters (6.1) with $a = 20$, $P = 10$, $\mu = 10$, $A = 0.2$ i.e. when the park is in imbalanced situation, if the habitat complexity is introduced but in a little amount (i.e. $c = 0.2$), it is seen from (Fig. 6.4(a)) that the density of tiger population is little higher than rhino population. That is the park is still in imbalanced situation. But if the value of habitat complexity increases i.e. $c = 0.5$, then the situation reverse (Fig. 6.4(b)) i.e. density of rhino is greater than density of tiger population. Thus the balanced behavior of the park is maintained by increasing the habitat complexity.

   ii. When $\mu < 1$

Similarly, introducing the habitat complexity ($c$) in the system (5.1) but in a small amount (i.e. $c = 0.2$), the park is showing ecologically imbalanced situation for the set of values of parameters (6.1) with $a = 20$, $P = 10$, $\mu = .4$, $A = 3$ (Fig. 6.5(a)). But if $c$ increases (i.e. $c = 0.5$), the park maintains the balanced behavior (6.5(b)).

Thus it is observed that if the attacking rate and poaching effect increase, the park is in an ecologically imbalanced situation. But, by introducing good quantity of additional food for the tiger population and/or habitat complexity on immature rhino population, ecologically balanced behavior can be maintained.

### 6.2. Dynamical behavior of the system

#### 6.2.1. Stability of the interior equilibrium point

To see the dynamical behavior of the system, the following set of values of the parameters

$$
\begin{aligned}
&a_1 = 45, \quad d_1 = 5, \quad a = 20, \quad h = 0.02, \quad r_1 = 22, \quad \rho = 2, \\
&d_2 = 0.3, \quad P = 10, \quad s = 1, \quad K = 0.05,
\end{aligned}
\tag{6.2}
$$

are chosen with the initial conditions $(1; 1; 1)$ and the positive equilibrium point $E_2(1.6668, 2.4214, 3.1980)$ is obtained. These set of parameters satisfy the local as well as global stability of conditions (3.4) and (3.8). The trajectories of $x_1, x_2$ and $y$ are plotted in Fig. 6.6 and it is observed from Fig. 6.6(a) that all the trajectories starting from $(1, 1, 1)$, first oscillate and then go to their respective equilibrium levels. Fig. 6.6(b) shows that the trajectory starts from the different initial point converges to the positive equilibrium point. This proves the local and global stability behavior of the system around the positive equilibrium point. It is also observed from Fig. 6.6(a) that as the attacking rate by the tiger population on immature rhino is high and due to the effect of poaching, the immature rhino and mature rhino go to the lower level and tiger population goes to higher equilibrium level.
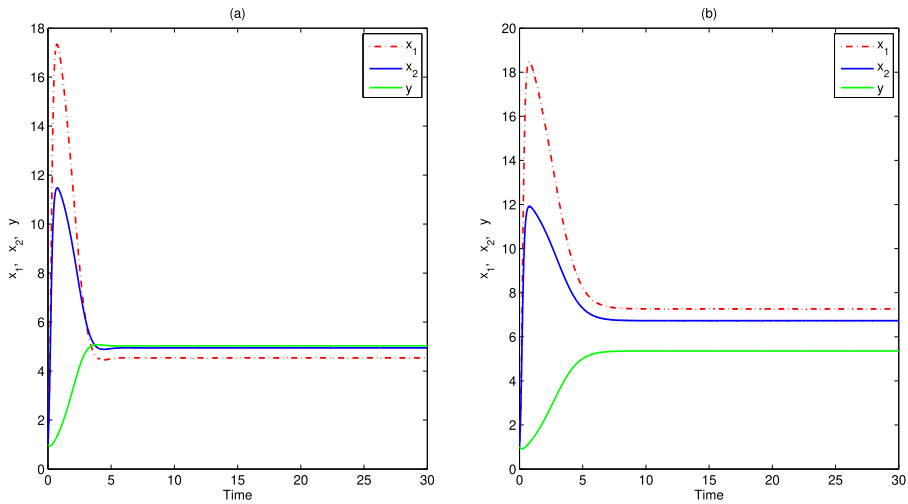
**Fig. 6.5.** Ecologically imbalanced and balanced behavior of the system (5.1) w.r.t $c$ for $\mu < 1$.
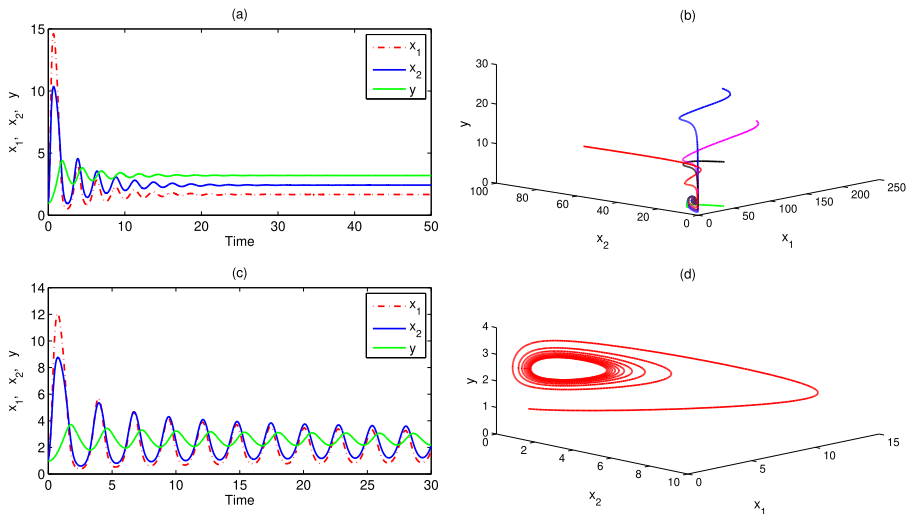


**Fig. 6.6.** (a) Local (b) Global stability of unique positive equilibrium (c) Oscillation behavior and (d) Limit cycle w.r.t $P = 12.45 > P^* = 12.15$.

### 6.2.2. Graph trajectory of immature rhino, mature rhino and tiger population considering poaching as a bifurcation parameter

The oscillation behavior of the rhino as well as the tiger population and also limit cycle through Hopf bifurcation are observed considering the poaching effect '$P$' as a bifurcation parameter. The critical point of poaching is $P = P^* = 12.15$. Thus, considering $P = 12.45 > P^*$ and other values same as (6.2), it is noticed that the trajectories starting from the initial point $(1,1,1)$, oscillate with respect to time $t$ (Fig. 6.6(c)) and also limit cycle around the positive equilibrium point (Fig. 6.6(d)). Thus the positive equilibrium point becomes unstable now. Thus to make it stable it will be better to concentrate the attention to control the poaching effect of the mature rhino.

Again if the attacking rate on the immature rhino by the tiger population increases then the positive equilibrium point $E_2$ becomes unstable. For the set of values of parameters (6.2) with initial point $(1,1,1)$ and $a = 25(> a^* = 23.5)$, the system shows oscillation behavior and limit cycle (Fig. 6.7(a–b)). Thus to control the attacking rate on immature rhino by the tiger population, the additional food is introduced in the second system.
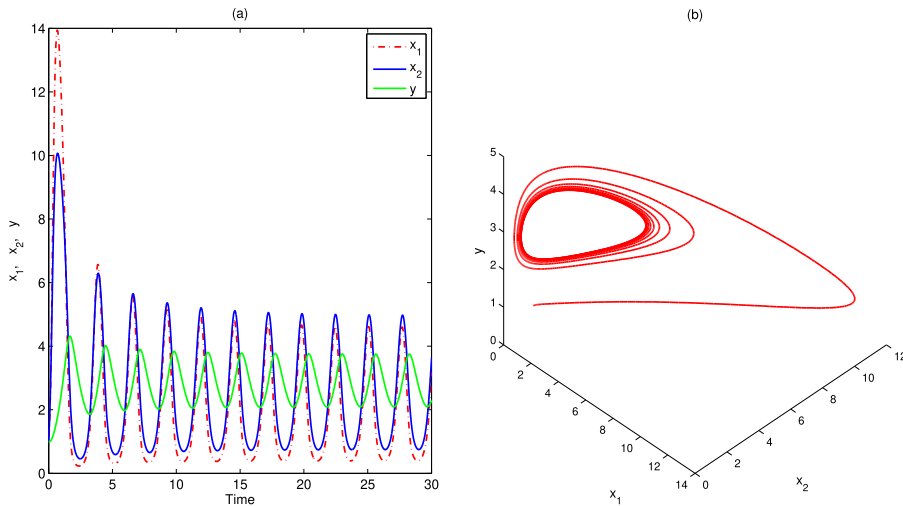
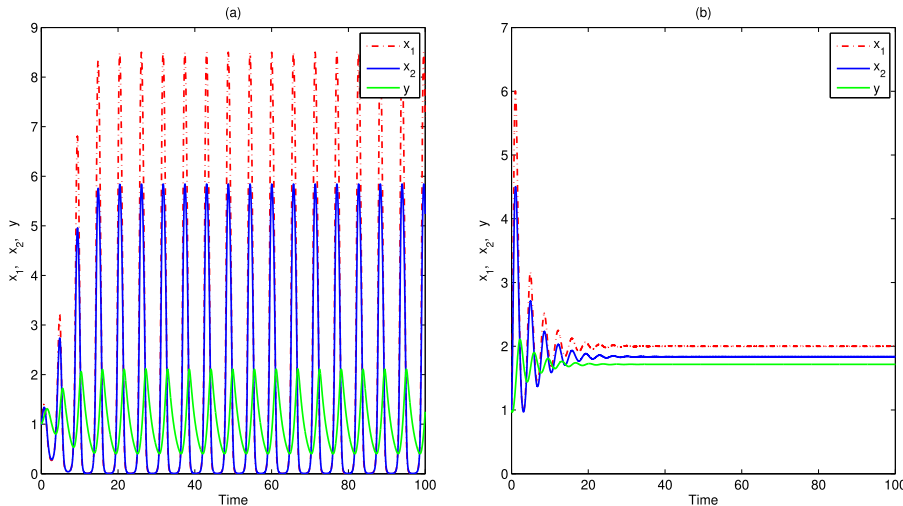**Fig. 6.7.** (a) Oscillatory behavior (b) Limit cycle of the system (3.2) w.r.t $a = 25 > 23.5 = a^*$.



**Fig. 6.8.** (a) Oscillatory behavior (b) Stable behavior of the positive equilibrium point w.r.t $a$.

*6.2.3. Graph trajectory of immature rhino, mature rhino and tiger population for different values of additional food quantity A*

In this section the system (4.1) has been studied with poaching on the mature rhino and additional food for the tiger population. We have chosen the set of values of parameters (6.2) with high attacking rate and poaching effect (i.e. $a = 25$, $P = 20$) i.e. the system is showing oscillation behavior. Now the quality and quantity of the additional food i.e. ($\mu = 10$, $A = 0.01$) is added to the system. Here it is observed that the equilibrium point remains unstable (Fig. 6.8(a)). But if the value of the quantity of the additional food increases (i.e. $A = 0.1$), then the positive equilibrium point becomes stable (Fig. 6.8(b)).

*6.2.4. Graph trajectory of immature rhino, mature rhino and tiger population for different values of habitat complexity c*

When the quantity of the additional food is less (i.e. Fig. 6.8(a)), to control the oscillation behavior, the habitat complexity is used. From (Fig. 6.9(a)), it is observed that when it is small i.e. $c = 0.1$, then the equilibrium point
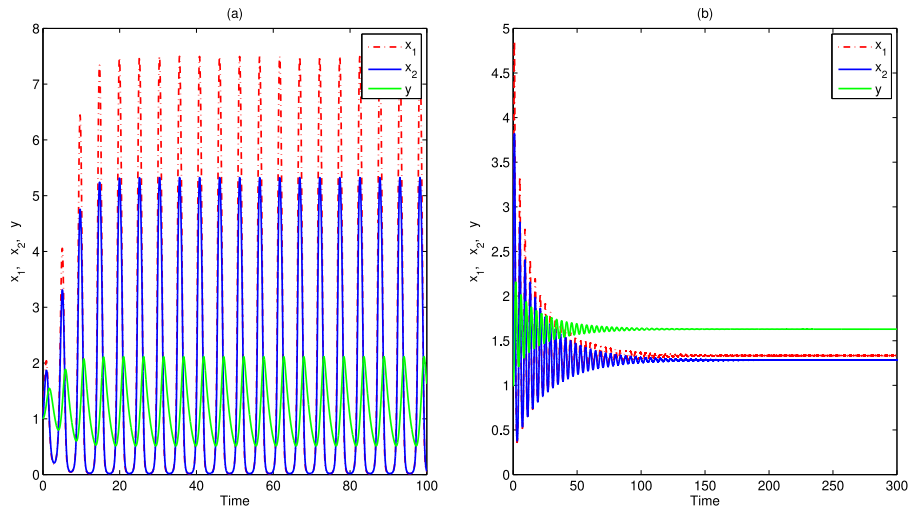
**Fig. 6.9.** (a) Oscillatory behavior (b) Stable behavior of the positive equilibrium point w.r.t $c$.
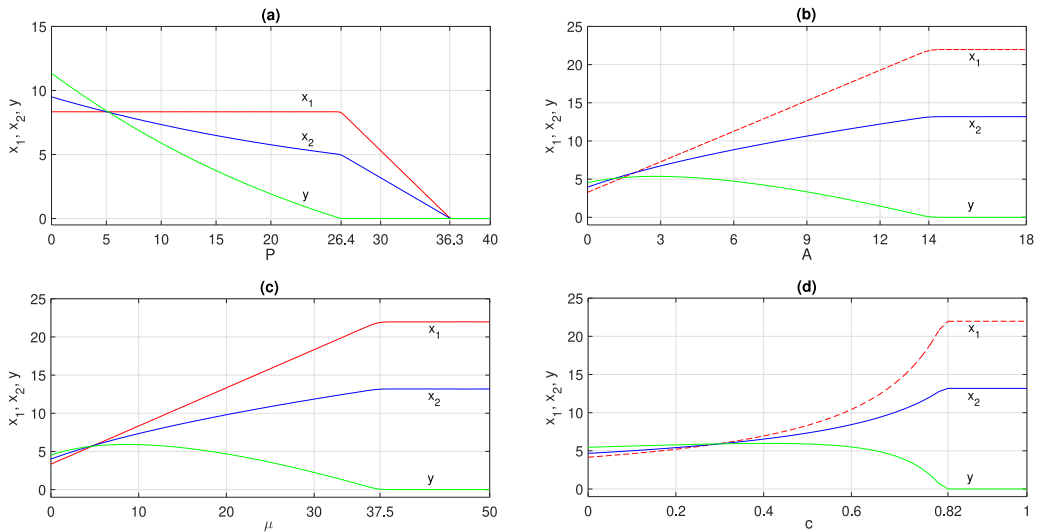


**Fig. 6.10.** Sensitivity for the state variables w.r.t $P$, $A$, $\mu$ and $c$.

still unstable but if c increases (i.e. $c = 0.4$), then stable behavior of the system can be seen (Fig. 6.9(b)). If the value of $c$ increases further (i.e. $c = 0.8$), the balanced behavior of the park is maintained.

### 6.3. Sensitivity analysis

The change of position of positive equilibrium due to the variation of the important parameters like poaching effect, quality and quantity of the additional food and habitat complexity are discussed in this section. For this discussion the following set of values of parameters are taken:

$$a_1 = 45, \quad d_1 = 5, \quad a = 20, \quad h = 0.02, \quad r_1 = 22, \quad \rho = 2,$$
$$d_2 = 0.3, \quad P = 10, \quad s = 1, \quad K = 0.05, \quad \mu = 10, \quad A = 0.2, \quad c = 0.5$$

with the initial condition $(1,1,1)$.

From Fig. 6.10(a), it is observed that the equilibrium of immature rhino ($x_1$) remains same up to P = 26.4. But equilibria of mature rhino ($x_2$) and tiger ($y$) population decrease slowly as the poaching effect increases. But at P =
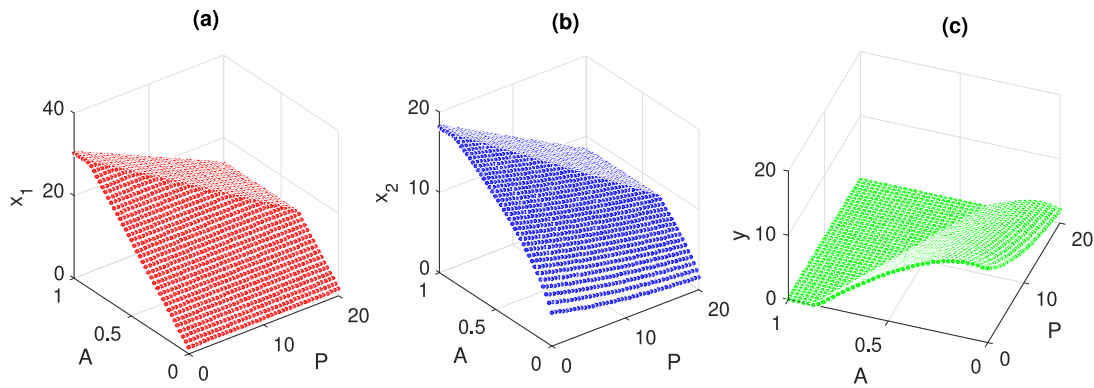
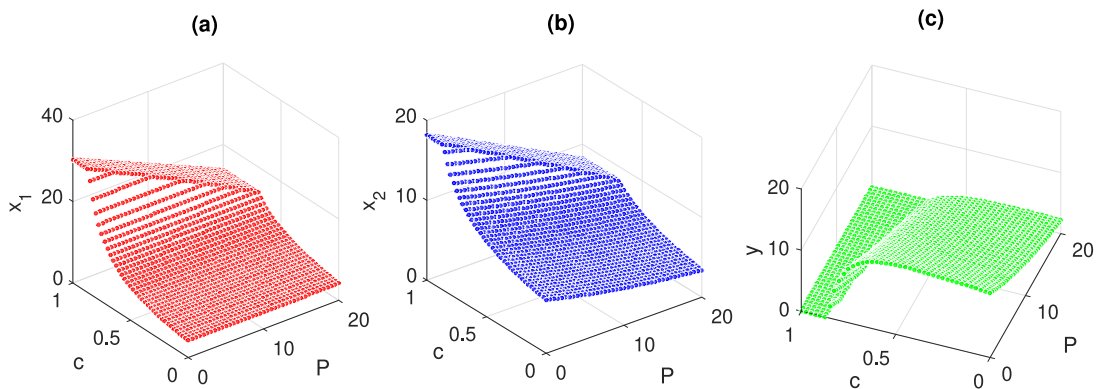**Fig. 6.11.** Distribution of equilibrium point w.r.t $(P, A)$.



**Fig. 6.12.** Distribution of equilibrium point w.r.t $(P, c)$.

26.4, the equilibrium points of both the rhino population drastically decrease and tiger population goes to zero level. For $P \geq 36.3$, the equilibria of all three population go to the extinction level. Fig. 6.10(b) describes the changes of the equilibrium with respect to quantity of the additional food '$A$' at $\mu = 0.4$. Here it is seen that initially the equilibrium of tiger population is larger than rhino population, but as A increases the equilibrium of tiger population decreases. Whereas the equilibria of both the rhino population increase with the increase of A. When $A \geq 14$, the equilibria of rhino remain same but the equilibrium of the tiger population becomes zero. Similar behavior is also observed in Fig. 6.10(c) (while varying '$\mu$' at A = 0.2) and Fig. 6.10(d) (with the variation of c).

Figs. 6.11 and 6.12 describe the distribution of the equilibrium point with the variation of the parameters (P, A) and (P, c) respectively. Here it is seen that as P and A increase, the equilibria of both the rhino population increase but equilibrium of tiger population decreases (Fig. 6.11). Similar behavior is also seen in Fig. 6.12.

## 7. Conclusion

According to the census report and real data collected from PCCF Wildlife, Assam, on Kaziranga National Park (KNP), three mathematical models are formulated and analyzed to understand the dynamics of rhino (prey) and tiger (natural enemy, the predator) interaction. As the tiger feeds on immature rhino, stage structure is considered on the rhino population by classifying them into two sub populations: immature rhino (horn not developed) and mature rhino (horn developed). In view of ecology, the main objective of this study is to increase the size of the rhinoceros population in this park and also to maintain the ecological balance of the park. From the mathematical point of view, the stability and complex dynamical behavior of the system are analyzed for all three models.

The first system describes the interaction between stage-structured rhino and tiger population in the presence of continuous threat of poaching. It is assumed that the tiger population consumes immature rhino only as per Holling type-II functional response. The intra-specific interaction between mature rhino is also considered. By considering the poaching effect (P) as a bifurcation parameter, the conditions(of existence of Hopf-bifurcation are established in this system. As the tiger preys on other herbivores of KNP too, the previous functional response is modified by taking additional food as another parameter. The analytical results describe the effect of additional food on the dynamics of the previous system.

The natural land-form provides certain degree of protection to the immature rhino from the tiger population. So the functional response of the second system is further modified by introducing habitat complexity. The previous analytical results are modified with the factor habitat complexity for this system. It may be noted that the origin is never stable in all three models, which implies that the system will not collapse for any values of parameters.

The numerical simulation is carried out for some biologically feasible values of parameters that satisfy various analytical conditions. This section is divided into three parts. The first part describes the ecologically balanced behavior of the park while the second discusses the complex dynamical behavior. The park is considered to be in a state of ecological balance if it maintains higher densities of the rhino population compared to tiger population. But, if the densities of rhino falls drastically it is ecologically imbalanced. It is observed that if the poaching effect and/or attacking rate are higher, then the park is in ecologically imbalanced situation. This imbalanced situation can be controlled by introducing appropriate amount of additional food and habitat complexity.

In the second part it is observed that if the poaching effect increases, then the positive equilibrium point becomes unstable showing its bifurcation behavior. Also, from the field data (during 1981–1996) it was observed that due to the increase of poaching effect there was a little fluctuation in the rhino population. The numerical simulation results are consistent with the real situation. Thus if the poaching rate is increased beyond a threshold value, an adverse effect on the rhino population is observed which could lead them towards extinction. Again, if the attacking rate on immature rhino by tiger population increases, the positive equilibrium point becomes unstable. To control the attacking rate and the oscillation behavior of the system, the additional food and habitat complexity become important parameters.

The change in the values of positive equilibrium with respect to parameters like poaching effect, additional food and habitat complexity are discussed in the third part. Initially, the equilibrium values of tiger population are found to be higher than rhino population. But at higher values of these parameters, the equilibrium of tiger goes to zero level. This may be because of the fact that in the presence of poaching and habitat complexity the tiger has to depend on additional food. However, the poor quantity and quality of additional food results in a drop in tiger population. Thus from the above discussion it may be concluded that the density of rhino and tiger population can be maintained in good number by introducing good quality and quantity of additional food and habitat complexity.

In this work, we have not taken any effort to control the poaching effect on mature rhino. In future research will be carried out by introducing some effort to control the poaching effect.

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